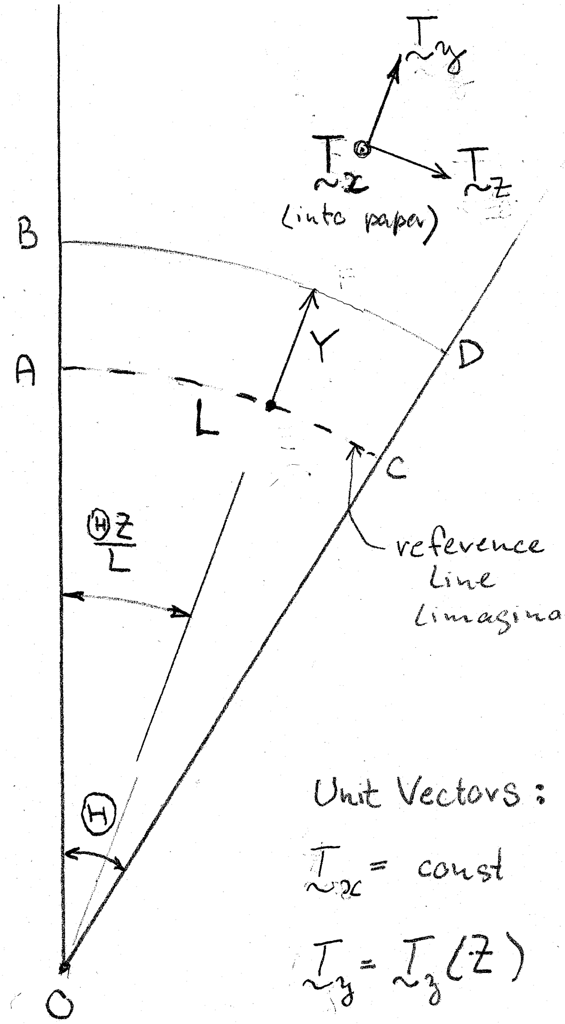
Generalized Plane Strain Formulation (melmt 13)

(X, Y, Z) Material Coordinates



Unit Vectors:

$\underset{\sim}{I}_x = \text{const}$

$\underset{\sim}{I}_y = \underset{\sim}{I}_y(Z)$

$\underset{\sim}{I}_z = \underset{\sim}{I}_z(Z)$

Unit Vectors

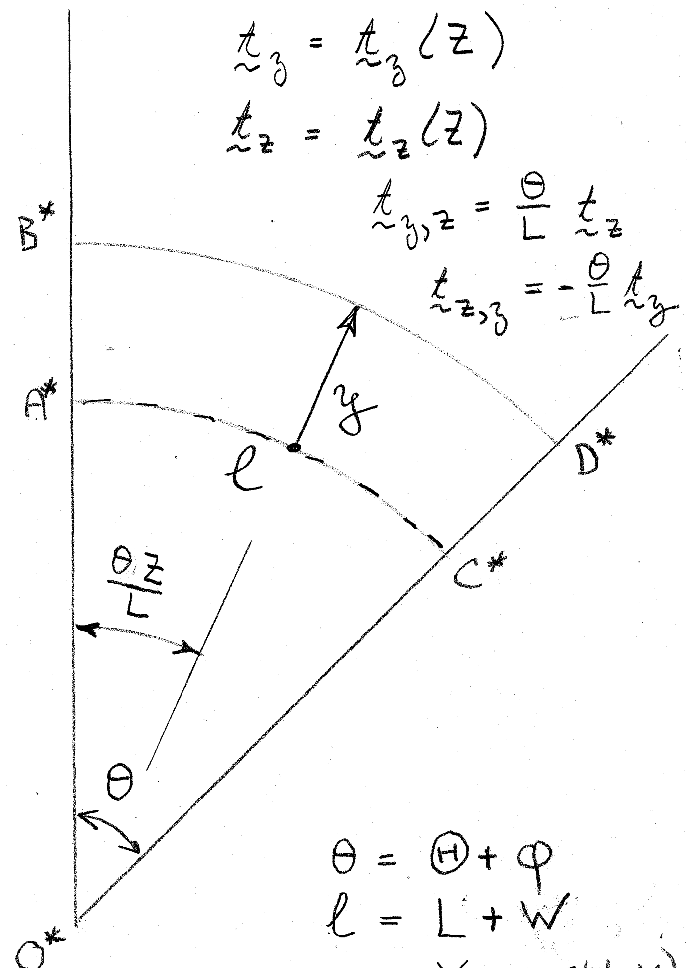
$\underset{\sim}{t}_x = \underset{\sim}{I}_x$ (const.)

$\underset{\sim}{t}_y = \underset{\sim}{t}_y(Z)$

$\underset{\sim}{t}_z = \underset{\sim}{t}_z(Z)$

$\underset{\sim}{t}_{y,z} = \frac{\theta}{L}\underset{\sim}{t}_z$

$\underset{\sim}{t}_{z,y} = -\frac{\theta}{L}\underset{\sim}{t}_y$



$\theta = \Theta + \varphi$

$\ell = L + W$

$y = Y + v(X, Y)$

$x = X + u(X, Y)$

Undeformed Configuration

$\underset{\sim}{I}_{y,z} = \frac{\Theta}{L}\underset{\sim}{I}_z \qquad \underset{\sim}{I}_{z,z} = -\frac{\Theta}{L}\underset{\sim}{I}_y$

$$\underset{\sim}{R} = X\underset{\sim}{I}_x + \left(Y + \frac{L}{\Theta}\right)\underset{\sim}{I}_y$$

$\underset{\sim}{R}_{,X} = \underset{\sim}{I}_X$

$\underset{\sim}{R}_{,Y} = \underset{\sim}{I}_Y$

$\underset{\sim}{R}_{,Z} = \left(Y + \frac{L}{\Theta}\right)\frac{\Theta}{L}\underset{\sim}{I}_Z$

$= \left(\frac{Y\Theta}{L} + 1\right)\underset{\sim}{I}_Z$

Deformed Configuration

$$\underset{\sim}{r} = x\underset{\sim}{t}_x + \left(y + \frac{\ell}{\theta}\right)\underset{\sim}{t}_y$$

$\underset{\sim}{r}_{,X} = x_{,X}\underset{\sim}{t}_x + v_{,X}\underset{\sim}{t}_y$

$\underset{\sim}{r}_{,Y} = y_{,Y}\underset{\sim}{t}_y + u_{,Y}$

$\underset{\sim}{r}_{,Z} = \left(y + \frac{\ell}{\theta}\right)\frac{\theta}{L}\underset{\sim}{t}_z$

$= \left(\frac{\ell}{} + \frac{y\theta}{}\right)\underset{\sim}{t}_z$

Undeformed Config

Length of fiber:

$$A_Z = L + Y\Theta$$

$$\underset{\sim}{G}_X = \underset{\sim}{R}_{,X} = \underset{\sim}{I}_X$$
$$\underset{\sim}{G}_Y = \underset{\sim}{R}_{,Y} = \underset{\sim}{I}_Y$$
$$\underset{\sim}{G}_Z = \underset{\sim}{R}_{,Z} = (A_Z/L)\,\underset{\sim}{I}_Z$$

$$\underset{\sim}{G}^X = \underset{\sim}{I}_X$$
$$\underset{\sim}{G}^Y = \underset{\sim}{I}_Y$$
$$\underset{\sim}{G}^Z = (L/A_Z)\,\underset{\sim}{I}_Z$$

Deformed Config

$$a_z = \ell + y\Theta$$

$$\underset{\sim}{g}_X = x_{,X}\,\underset{\sim}{t}_x + v_{,X}\,\underset{\sim}{t}_y$$
$$\underset{\sim}{g}_Y = y_{,Y}\,\underset{\sim}{t}_y + u_{,Y}\,\underset{\sim}{t}_x$$
$$\underset{\sim}{g}_Z = (a_z/L)\,\underset{\sim}{t}_z$$

$$\underset{\sim}{g}^Z = (L/a_z)\,\underset{\sim}{t}_z$$

Def. Gradient Tensor

$$\underset{\sim}{F} = \underset{\sim}{r}_{,X} \underset{\sim}{I}_X + \underset{\sim}{r}_{,Y} \underset{\sim}{I}_Y + \underset{\sim}{r}_{,Z} \, (L/A_z) \, \underset{\sim}{I}_Z$$

Green Lagr. Strain Tensor : $(2\underset{\sim}{E} = \underset{\sim}{F}^T \cdot \underset{\sim}{F} - \underset{\sim}{1})$

$$\underset{\sim}{E} = E_{XX} \underset{\sim}{I}_X \underset{\sim}{I}_X + E_{YY} \underset{\sim}{I}_Y \underset{\sim}{I}_Y + E_{XY} (\underset{\sim}{I}_X \underset{\sim}{I}_Y + \underset{\sim}{I}_Y \underset{\sim}{I}_X) + E_{ZZ} \underset{\sim}{I}_Z \underset{\sim}{I}_Z$$

$$E_{XX} = u_{,X} + \frac{1}{2}(u_{,X}^2 + v_{,X}^2)$$

$$E_{YY} = v_{,Y} + \frac{1}{2}(u_{,Y}^2 + v_{,Y}^2)$$

$$2\,E_{XY} = u_{,Y} + v_{,X} + u_{,X} u_{,Y} + v_{,X} v_{,Y}$$

$$E_{ZZ} = \frac{1}{2}\left(1 + \frac{\Theta}{L} Y\right)^{-2} \left\{ \underset{\sim}{r}_{,Z} \cdot \underset{\sim}{r}_{,Z} - \left(1 + \frac{\Theta}{L} Y\right)^2 \right\}$$

$$E_{ZZ} = \frac{1}{2}(L + \Theta Y)^{-2}\left\{(\ell + \theta y)^2 - (L + \Theta Y)^2\right\}$$

$$\text{Let } \begin{cases} A_z = L + \Theta Y \ (= BD) \\ a_z = \ell + \theta y \ (= B^* D^*) \end{cases} \qquad E_{ZZ} = \frac{1}{2}\left(\frac{a_z^2 - A_z^2}{A_z^2}\right)$$

$$\begin{aligned} a_z &= L + W + (\Theta + \varphi)(Y + v) \\ &= L + \Theta Y + W + \Theta v + \varphi(Y + v) \\ &= A_z + W + \Theta v + \varphi y \end{aligned}$$

$$\boxed{\begin{aligned} &\Rightarrow \quad E_{ZZ} = \epsilon_z + \frac{1}{2}\epsilon_z^2, \quad \epsilon_z = \frac{W + \Theta v + \varphi y}{A_z} \\ &A_z = L + \Theta Y \\ &dV = A_z \, dX \, dY \quad \text{(undeformed volume element)} \end{aligned}}$$

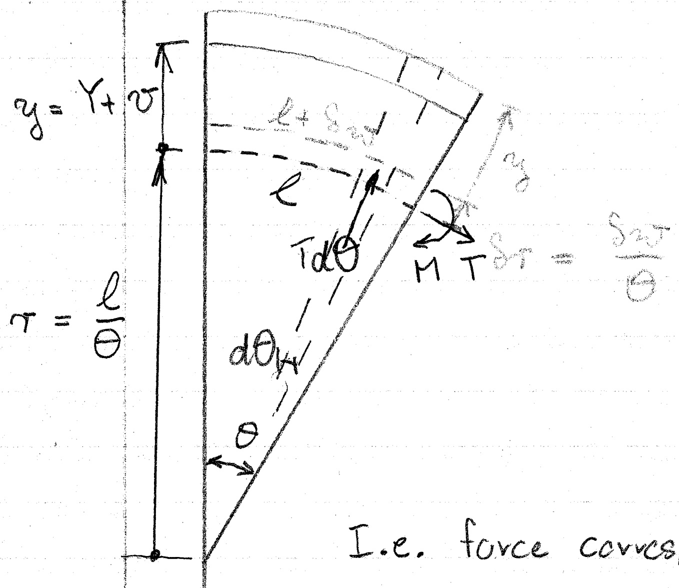
$$\delta E_{ZZ} = \frac{a_z}{A_z^2}\,\delta a_z = \frac{a_z}{A_z^2}(\delta W + \delta\varphi\, y + \theta\,\delta v)$$

$$\begin{aligned} \delta_1 \delta_2 E_{ZZ} &= \frac{1}{A_z^2}(\delta_1 a_z \delta_2 a_z + a_z\, \delta_1 \delta_2 a_z) \\ &= \frac{1}{A_z^2}\Big[(\delta_1 w + y\,\delta_1\varphi + \theta\,\delta_1 v)(\delta_2 w + y\,\delta_2\varphi + \theta\,\delta_2 v) \\ &\qquad + a_z(\delta_1 v\, \delta_2\varphi + \delta_2 v\, \delta_1\varphi)\Big] \end{aligned}$$

To interpret the applied forces corresponding to w & φ 4
Consider EVW for system of External Applied Forces in Equilibrium for virtual displacements δw & $\delta\varphi$:

Virtual Displ δw $\quad l = L + w \Rightarrow \delta l = \delta w$

$\Theta = \Theta + \varphi \Rightarrow \delta\Theta = \delta\varphi$



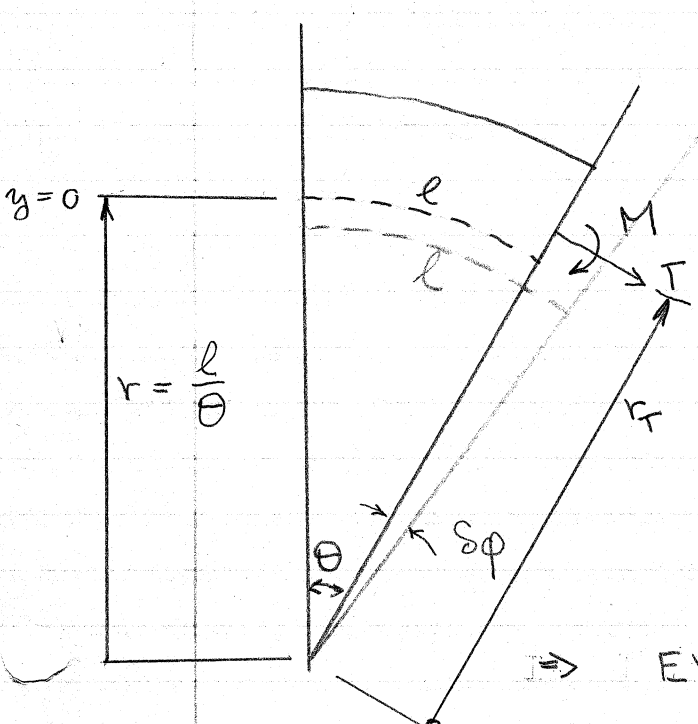
Ext. Virt. Wk.

$$EVW = \frac{\delta w}{\Theta}\int T d\Theta = T\delta w$$

Note: for syst in equil., rigid body displ & rot. may always be added in evaluating Ext. Virt. Wk.

I.e. force corresponding to w is the axial tension.

Virt. Displ. $\delta\varphi$ $\quad \delta\Theta = \delta\varphi, \quad \delta r = -\frac{l\,\delta\varphi}{\Theta^2}$



$$EVW = M\,\delta\varphi$$
$$+ T\,r_T\,\delta\varphi$$
$$- \frac{l\,\delta\varphi}{\Theta^2}\int T d\Theta$$

$$EVW = \{M + T(r_T - r)\}\,\delta\varphi$$

where $r_T = r + y_T$

$$\Rightarrow EVW = (M + y_T T)\,\delta\varphi$$

[y_T = y coordinate of point where ...

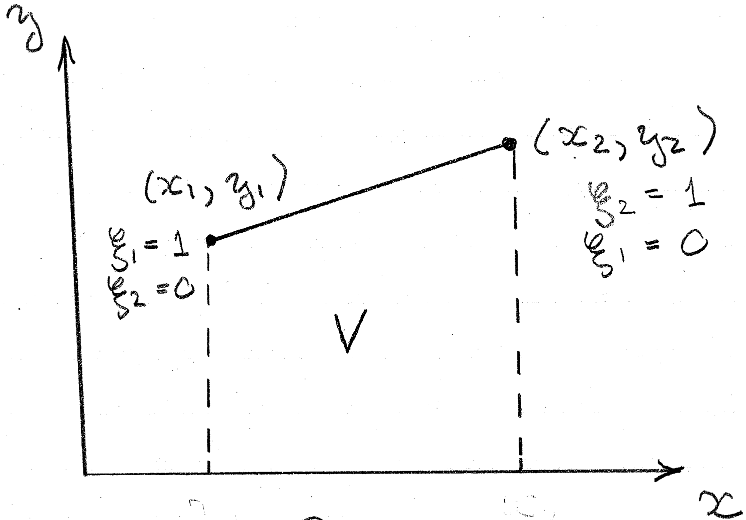
Pressure Loads Incl End Caps

With ref. to p. 1 the area $A^*C^*D^*B^*$ is

$$A = \frac{1}{2}\theta\left[(O^*B^*)^2 - (O^*A^*)^2\right]$$

$$= \frac{1}{2}\theta\left[(r+z)^2 - r^2\right] \quad , \quad r = O^*A^*$$

$$= \theta r z + \frac{1}{2}\theta z^2 = \ell z + \frac{1}{2}\theta z^2$$



$$V = \int_{①}^{②} A\, dx \quad , \quad \begin{aligned} x &= \xi_1 x_1 + \xi_2 x_2 \\ z &= \xi_1 z_1 + \xi_2 z_2 \\ \xi_1 + \xi_2 &= 1 \\ \xi_2 &= \xi \quad , \text{ say} \end{aligned}$$

$$V = \int_0^1 \left(\ell z + \frac{1}{2}\theta z^2\right) x_{12}\, d\xi \quad , \quad x_{12} = x_2 - x_1$$

$$= \int_0^1 \left\{ \ell\left[\xi_1 z_1 + \xi_2 z_2\right] + \frac{1}{2}\theta\left[\xi_1 z_1 + \xi_2 z_2\right]^2 \right\} x_{12}\, d\xi_2$$

$$= \left\{ \frac{1}{2}\ell(z_1 + z_2) + \frac{1}{6}\theta(z_1^2 + z_1 z_2 + z_2^2) \right\} x_{12}$$

Let $\quad y_{av} = \frac{1}{2}(y_1 + y_2)$

$$y_{ms} = \frac{1}{3}(y_1^2 + y_1 y_2 + y_2^2)$$

$$\frac{\partial y_{av}}{\partial y_\alpha} = \frac{1}{2} \qquad (\alpha = 1,2)$$

$$\frac{\partial y_{ms}}{\partial y_\alpha} = b_\alpha, \quad b_1 = \frac{1}{3}(2y_1 + y_2), \quad b_2 = \frac{1}{3}(2y_2 + y_1)$$

$$V = \left(\ell\, y_{av} + \frac{1}{2}\,\theta\, y_{ms}\right) x_{12}$$

$$\frac{\partial V}{\partial x_{12}} = \ell\, y_{av} + \frac{1}{2}\,\theta\, y_{ms}$$

$$\frac{\partial V}{\partial y_\alpha} = \frac{1}{2}\left(\ell + \theta\, b_\alpha\right) x_{12}$$

$$\frac{\partial V}{\partial \ell} = y_{av}\, x_{12}$$

$$\frac{\partial V}{\partial \theta} = \frac{1}{2}\, y_{ms}\, x_{12}$$

Let $\underline{x} = [\,x_{12} \quad y_1 \quad y_2 \quad \ell \quad \theta\,]^T$

$$\frac{\partial V}{\partial \underline{x}} = \begin{bmatrix} \ell\, y_{av} + \frac{1}{2}\theta\, y_{ms} \\ \frac{1}{2}(\ell + \theta b_1)\, x_{12} \\ \frac{1}{2}(\ell + \theta b_2)\, x_{12} \\ y_{av}\, x_{12} \\ \frac{1}{2} y_{ms}\, x_{12} \end{bmatrix}$$

$$\frac{\partial^2 V}{\partial \underline{x}\, \partial \underline{x}^T} = \begin{bmatrix} 0 & \frac{1}{2}(\ell+\theta b_1) & \frac{1}{2}(\ell+\theta b_2) & y_{av} & \frac{1}{2} y_{ms} \\ \frac{1}{2}(\ell+\theta b_1) & \frac{1}{3}\theta x_{12} & \frac{1}{6}\theta x_{12} & \frac{1}{2}x_{12} & \frac{1}{2} b_1 x_{12} \\ \frac{1}{2}(\ell+\theta b_2) & \frac{1}{6}\theta x_{12} & \frac{1}{3}\theta x_{12} & \frac{1}{2}x_{12} & \frac{1}{2} b_2 x_{12} \\ y_{av} & \frac{1}{2}x_{12} & \frac{1}{2}x_{12} & 0 & 0 \\ \frac{1}{2} y_{ms} & \frac{1}{2} b_1 x_{12} & \frac{1}{2} b_2 x_{12} & 0 & 0 \end{bmatrix}$$

E.g. for $x_{12} = 4$, $y_1 = 5$, $y_2 = 8$, $\theta = .3$, $\ell = 10$,
(ref ma25)

$$\frac{\partial V}{\partial \underline{x}} = \begin{bmatrix} 71.45 \\ 23.6 \\ 24.2 \\ 26 \\ 86 \end{bmatrix}$$

$$\frac{\partial^2 V}{\partial \underline{x} \partial \underline{x}^T} = \begin{bmatrix} 0 & 5.9 & 6.05 & 6.5 & 21.5 \\ 5.9 & 0.4 & 0.2 & 2 & 12 \\ 6.05 & 0.2 & 0.4 & 2 & 14 \\ 6.5 & 2 & 2 & 0 & 0 \\ 21.5 & 12 & 14 & 0 & 0 \end{bmatrix}$$

For positive pressure p acting from left to right when looking from node 1 towards node 2, the potential of the pressure load is

$$\Pi = pV$$

$$\underline{p}^{(e)} = -\Pi_{,q} = -p \begin{bmatrix} -V_{,x_{12}} \\ V_{,y_1} \\ V_{,x_{12}} \\ V_{,y_2} \\ V_{,\ell} \\ V_{,\theta} \end{bmatrix} \qquad \underline{k}^{(e)} = p \begin{bmatrix} V_{,x_{12}x_{12}} & -V_{,x_{12}y_1} & -V_{,x_{12}x_{12}} \\ -V_{,y_1x_{12}} & V_{,y_1y_1} & V_{,y_1x_{12}} \\ -V_{,x_{12}x_{12}} & V_{,x_{12}y_1} & V_{,x_{12}x_{12}} \\ -V_{,y_2x_{12}} & V_{,y_1y_1} & V_{,y_2x_{12}} \\ -V_{,\ell x_{12}} & V_{,\ell y_1} & V_{,\ell x_{12}} \\ -V_{,\theta x_{12}} & V_{,\theta y_1} & V_{,\theta x_{12}} \end{bmatrix} \text{ etc.}$$

$$\underline{q} = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & W & \varphi \end{bmatrix}^T$$

Body Force

Force per unit undeformed volume:

$$\underset{\sim}{b} = b_x \underset{\sim x}{t} + b_y \underset{\sim y}{t} + b_z \underset{\sim z}{t} \quad . \text{ Assume } b_z = 0.$$

Virtual Displ.

$$\delta \underset{\sim}{u} = \delta u \, \underset{\sim x}{t} + \left(\delta v + \frac{\delta w}{\theta} - \frac{\ell}{\theta^2} \delta\varphi \right) \underset{\sim y}{t}$$

$$+ \left(y + \frac{\ell}{\theta} \right) \underset{\sim z}{t} \, \frac{z}{L} \, \delta\varphi$$

Virt. Work per unit undeformed Volume:

$$\delta W_e = \underset{\sim}{b} \cdot \delta \underset{\sim}{u}$$
$$= b_x \, \delta u + b_y \left(\delta v + \frac{\delta w}{\theta} - \frac{\ell \, \delta\varphi}{\theta^2} \right)$$

$$dV = A_z \, dX \, dY \qquad A_z = L + \Theta Y$$

$$= A_z \, \mathrm{Det}\left(\frac{d\underset{\sim}{X}}{d\underset{\sim}{\xi}^T} \right) d\xi \, d\eta$$

Total Ext. Virt Wk.

$$EVW = \int_{x,y} (b_x \, \delta u + b_y \, \delta v) A_z \, dx \, dy$$
$$+ \left(\frac{\delta w}{\theta} - \frac{\ell}{\theta^2} \delta\varphi \right) \int_{x,y} b_y \, A_z \, dx \, dy$$

Leave out 2nd term. At most this affects the interpretation of the nodal forces corresponding to w & φ. Indeed omitting this term, it remains that these must be the axial tension T acting at $y=0$ and moment M, from p.4, respectively. Thus use

$$EVW = \int (b_x \, \delta u + b_y \, \delta v) A_z \, \mathrm{Det}(J) \, d\xi \, d\xi$$
$$\cong \sum_{i=1}^{4} \left(b_x \, \delta u_i + b_y \, \delta v_i \right) A_z \, \mathrm{Det}(J)$$

for single int. pt. in middle