Implementation of Riks Arclength Method in npex Program

Equation to be solved:

$$\underline{f}(\underline{u}, \lambda) = \underline{0} \qquad \lambda \text{ specified.}$$

Instead, solve:

$$\underline{F}(\underline{u}, \lambda, \eta) \equiv \begin{Bmatrix} \underline{f}(\underline{u}, \lambda) \\ h(\underline{u}, \lambda, \eta) \end{Bmatrix} = \underline{0}$$

or $\underline{F}(\underline{z}, \eta) = 0$ where $\underline{z} = (\underline{u}, \lambda)$

η specified

Let $\underline{z}_0 = (\underline{u}_0, \lambda_0)$ = converged solution, or init guess

$$\dot{\underline{z}}_0 \equiv \left.\frac{d\underline{z}}{d\eta}\right|_{\eta=\eta_0} = (\dot{\underline{u}}_0, \dot{\lambda}_0)$$ = computed rate from previous loadstep.

$$h(\underline{u}, \lambda, \eta) \equiv \frac{1}{\alpha}\left\{ \dot{\underline{u}}_0^T \underline{W}(\underline{u} - \underline{u}_0) + \dot{\lambda}_0 W_\lambda (\lambda - \lambda_0) \right\} - (\eta - \eta_0)$$

where $\underline{W}$, W_λ = weighting matrices that define inner product in load-displacement, $\underline{z} = (\underline{u}, \lambda)$ space.

Partition unknown displacements & corresponding forces as

$$\underline{u} = (\hat{\underline{u}}, u)$$

$$\underline{f} = (\hat{\underline{f}}, f)$$

will be referred to as pivotal displacement

Consider Solution of

$$\underline{F}_{,\underline{z}} \, \Delta \underline{z} = -\underline{F}$$

$$\Rightarrow \quad \hat{\underline{f}}_{,\hat{\underline{u}}} \, \Delta\hat{\underline{u}} + \hat{\underline{f}}_{,u} \, \Delta u + \hat{\underline{f}}_{,\lambda} \, \Delta\lambda = -\hat{\underline{f}}$$

$$f_{,\hat{\underline{u}}} \, \Delta\hat{\underline{u}} + f_{,u} \, \Delta u + f_{,\lambda} \, \Delta\lambda = -f$$

$$h_{,\hat{\underline{u}}} \, \Delta\hat{\underline{u}} + h_{,u} \, \Delta u + h_{,\lambda} \, \Delta\lambda = -h$$

This is of form
$$\begin{bmatrix} \underline{G}_{11} & \underline{G}_{12} \\ \underline{G}_{21} & \underline{G}_{22} \end{bmatrix} \begin{bmatrix} \Delta\hat{\underline{u}} \\ \Delta\check{\underline{u}} \end{bmatrix} = \begin{bmatrix} -\hat{\underline{f}} \\ -\check{\underline{f}} \end{bmatrix}$$

$$\Rightarrow \underline{A} \, \Delta\check{\underline{u}} = -(\check{\underline{f}} - \underline{G}_{21} \underline{G}_{11}^{-1} \hat{\underline{f}}) = -\left\{ \begin{bmatrix} f \\ h \end{bmatrix} - \begin{bmatrix} f_{,\hat{\underline{u}}} \\ h_{,\hat{\underline{u}}} \end{bmatrix} (\hat{\underline{f}}_{,\hat{\underline{u}}})^{-1} \hat{\underline{f}} \right\}$$

where $\underline{A} = \underline{G}_{22} - \underline{G}_{21} \underline{G}_{11}^{-1} \underline{G}_{12}$

$$A_{11} = f_{,u} - f_{,\hat{\underline{u}}} (\hat{\underline{f}}_{,\hat{\underline{u}}})^{-1} \hat{\underline{f}}_{,u}$$

$$A_{12} = f_{,\lambda} - f_{,\hat{\underline{u}}} (\hat{\underline{f}}_{,\hat{\underline{u}}})^{-1} \hat{\underline{f}}_{,\lambda}$$

$$A_{21} = h_{,u} - h_{,\hat{\underline{u}}} (\hat{\underline{f}}_{,\hat{\underline{u}}})^{-1} \hat{\underline{f}}_{,u}$$

$$A_{22} = h_{,\lambda} - h_{,\hat{\underline{u}}} (\hat{\underline{f}}_{,\hat{\underline{u}}})^{-1} \hat{\underline{f}}_{,\lambda}$$

$$\underline{G}_{11} \; \Delta \hat{\underline{u}} = - \hat{\underline{f}} - \underline{G}_{12} \; \Delta \check{\underline{u}}$$

$$\Rightarrow \Delta \hat{\underline{u}} = \left(\hat{\underline{f}}_{,\hat{\underline{u}}} \right)^{-1} \left\{ - \hat{\underline{f}} - \hat{\underline{f}}_{,u} \Delta u - \hat{\underline{f}}_{,\lambda} \Delta \lambda \right\}$$

For the case of the rate equation (step 7)
$\hat{\underline{f}}$, f and h are replaced by
$\underline{0}$, 0 and -1 respectively, giving

$$\underline{A} \begin{bmatrix} \dot{u} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\& \quad \dot{\hat{\underline{u}}} = \left(\hat{\underline{f}}_{,\hat{\underline{u}}} \right)^{-1} \left\{ - \hat{\underline{f}}_{,u} \dot{u} - \hat{\underline{f}}_{,\lambda} \dot{\lambda} \right\}$$

Procedure

0) Initialize

$\underline{u}_0 = 0 \qquad \lambda_0 = 0$ (or user spec.)

$\dot{\underline{u}}_0 = 0 \qquad \dot{\lambda}_0 = 1/\sqrt{W_\lambda}$ (first step fully Load controlled)

1) Initial Estimate

$\underline{u} = \underline{u}_0 + \Delta\eta \; \dot{\underline{u}}_0$

$\lambda = \lambda_0 + \underbrace{\Delta\eta}_{\text{loadstep increment}} \; \dot{\lambda}_0$

{ for 2nd & higher load step can use improved cubic predictor.

2) Calculate

$h_{,\underline{u}} \equiv \dot{\underline{u}}_0^T \underline{W}$

$h_\lambda = \dot{\lambda}_0 W_\lambda$

Note that

$$h = h_{,\underline{u}} (\underline{u} - \underline{u}_0) + h_\lambda (\lambda - \lambda_0) - \Delta\eta$$

is zero for the initial estimate used here, and need therefore not be calculated. Since it is a linear function, it is always zero in the iteration loop.

3) Calculate $\underline{f}(\underline{u}, \lambda)$

4) Form & factor $\underline{F}_{,z}$ (as on p. 2)

5) Solve System

$$\underline{F}_{,z} \, \Delta \underline{z} = - \underline{F}$$

as indicated on p. 2

6) Check convergence, if converged go to step. 7
Else
Update
$$\underline{u} \rightarrow \underline{u} + \Delta \underline{u}$$
$$\lambda \rightarrow \lambda + \Delta \lambda$$
$(\underline{z} \rightarrow \underline{z} + \Delta \underline{z})$

Repeat Steps 3 - 6 until convergence.

7) Load step completed.

$$\underline{F}_{,z} \, \dot{\underline{z}} = - \underline{F}_{,\eta}$$

$$\Rightarrow \quad \dot{\underline{u}}_0 \quad \& \quad \dot{\lambda}_0$$ (as described on p. 2a)

Normalize $\dot{\underline{u}}_0 = \frac{1}{\alpha} \dot{\underline{u}}_0 \qquad \dot{\lambda}_0 = \frac{1}{\alpha} \dot{\lambda}_0$

where $\alpha = \left\{ \dot{\underline{u}}_0^T \, \underline{W} \, \dot{\underline{u}}_0 + W_\lambda \, \dot{\lambda}_0^2 \right\}^{1/2}$

Set $\underline{u}_0 = \underline{u} \quad , \qquad \lambda_0 \equiv \lambda$

& go to Step 1

Path Parameters & Their Derivatives

Angle between path and plane λ = const

$$\sin\theta = \frac{\dot{\lambda}}{\lambda_1} \qquad \lambda_1 = 1/\sqrt{W_\lambda} = \text{unit load}$$

$$\Rightarrow \dot{\theta}\cos\theta = \frac{\ddot{\lambda}}{\lambda_1} \qquad \Rightarrow \ddot{\theta} = \frac{1}{\cos\theta}\,\frac{\ddot{\lambda}}{\lambda_1}$$

Curvature of Path

$$R = \|\ddot{z}\| = \left\{ W_\lambda \ddot{\lambda}^2 + \underline{u}^T \underline{W}_u \underline{u} \right\}^{1/2}$$

$$\geqslant \left\{ W_\lambda \ddot{\lambda}^2 \right\}^{1/2} = \left| \ddot{\lambda} / \lambda_1 \right|$$

Assume $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \cos\theta > 0$

$$|\ddot{\theta}| = \frac{1}{\cos\theta}\left|\frac{\ddot{\lambda}}{\lambda_1}\right| \leq \frac{R}{\cos\theta}$$

Number of Negative Eigenvalues of $\underline{f}_{,\underline{u}}$

Consider $\underline{K} = \underline{f}_{,\underline{u}} = \begin{bmatrix} \underline{K}_{11} & \underline{K}_{12} \\ \underline{K}_{12}^T & K_{22} \end{bmatrix}$

$\underline{K} = \underline{L}\,\underline{D}\,\underline{L}^T$, $\underline{L}$ = lower triangular

$\underline{L} = \begin{bmatrix} \underline{L}_{11} & \cdot \\ \underline{L}_{21} & 1 \end{bmatrix}$ $\quad \underline{D} = \begin{bmatrix} \underline{D}_{11} & \\ & D_{22} \end{bmatrix}$

wish to show that

$D_{22} = K_{22} - \underline{K}_{12}^T\,\underline{K}_{11}^{-1}\,\underline{K}_{12} = A_{11}$ (as defined on p.2)

$\underline{K} = \underline{L}\,\underline{D}\,\underline{L}^T = \underline{L}\begin{bmatrix} \underline{D}_{11} & \\ & D_{22} \end{bmatrix}\begin{bmatrix} \underline{L}_{11}^T & \underline{L}_{21}^T \\ & 1 \end{bmatrix}$

$= \begin{bmatrix} \underline{L}_{11} & \\ \underline{L}_{21} & 1 \end{bmatrix}\begin{bmatrix} \underline{D}_{11}\underline{L}_{11}^T & \underline{D}_{11}\underline{L}_{21}^T \\ & D_{22} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \underline{K}_{11} & \underline{K}_{12} \\ \underline{K}_{12}^T & K_{22} \end{bmatrix} = \begin{bmatrix} \underline{L}_{11}\underline{D}_{11}\underline{L}_{11}^T & \underline{L}_{11}\underline{D}_{11}\underline{L}_{21}^T \\ \underline{L}_{21}\underline{D}_{11}\underline{L}_{11}^T & \underline{L}_{21}\underline{D}_{11}\underline{L}_{21}^T + D_{22} \end{bmatrix}$

$(1,2) \Rightarrow \underline{L}_{11}^{-1}\underline{K}_{12} = \underline{D}_{11}\underline{L}_{21}^T$, $(2,1) \Rightarrow \underline{K}_{12}^T\,\underline{L}_{11}^{-1\,T}\,\underline{D}_{11}^{-1} = \underline{L}_{21}$

$\Rightarrow K_{22} = D_{22} + \underline{L}_{21}\underline{L}_{11}^{-1}\underline{K}_{12}$

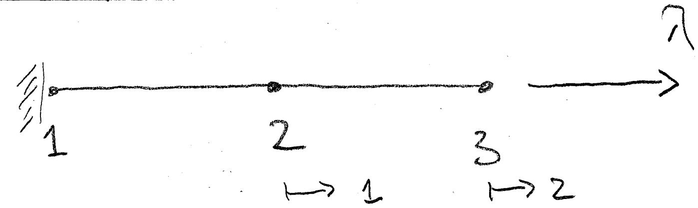
$= D_{22} + \underline{K}_{12}^T\,(\underline{L}_{11}^{-1})^T\,\underline{D}_{11}^{-1}\,\underline{L}_{11}^{-1}\,\underline{K}_{12}$

$= D_{22} + \underline{K}_{12}^T\,\underline{K}_{11}^{-1}\,\underline{K}_{12}$ (which proves result)

$\Rightarrow \quad A_{11} > 0 \Rightarrow NNEV(\underline{f}_{,\underline{u}}) = NNEV(\underline{f}_{,\hat{\underline{u}}})$

$A_{11} < 0 \Rightarrow NNEV(\underline{f}_{,\underline{u}}) = NNEV(\hat{\underline{f}}_{,\hat{\underline{u}}}) + 1$

Example



$F = e - e^3$

$F' = 1 - 3e^2$

$F' = 0$ at $e = \frac{1}{\sqrt{3}}$

$F_{max} = .38490$

WFU = WFL = 1

First loadstep : $t = 0.1$

$u_2 = e$

iteration	e	F	$0.1-F$	F'	Δe
1	0	0	0.1	1	0.1
2	0.1	.099	.001	.97	.001031
3	.101031	.100000	3.19×10^{-7}	.969	3.2×10^{-7}

Full-calculation for iteration 2 :

$$\underline{u} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad \lambda = 0.1, \quad \underline{f}(\underline{u}, \lambda) = \begin{bmatrix} 0 \\ .001 \end{bmatrix}$$

$$\underline{f}_{,\underline{u}} = \begin{bmatrix} 1.94 & -.97 \\ -.97 & .97 \end{bmatrix} \qquad \begin{matrix} \hat{u} \\ u \end{matrix}$$

$$A_{11} = .97 - (-.97)(1.94)^{-1}(-.97) = .485$$

$$A_{12} = 1 - (-.97)(1.94)^{-1}(0) = 1$$

$$A_{21} = 0$$

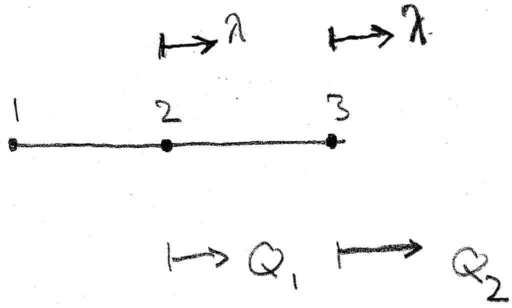
$$A_{22} = 1$$

$$\underline{b} = \begin{bmatrix} -.001 & + & (.97)(1.94)^{-1}(0) \\ 0 & + & 0 \end{bmatrix} = \begin{bmatrix} -.001 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.48500 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} -0.001 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \Delta u \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} .002062 \\ 0 \end{bmatrix}$$

Example Problem



$$F = 10e - \tfrac{1}{3}1000e^3$$
$$= 10e\left(1 - \tfrac{1}{3}100e^2\right)$$
$$F = 0 \text{ at } e = 0.1\sqrt{3}$$
$$F' = 10 - 1{,}000\, e^2$$
$$F' = 0 \text{ @ } e = 0.1$$
$$\Rightarrow F = 10\,\frac{1}{10} - \frac{1}{3}\,\frac{10^3}{10^3}$$
$$= \frac{2}{3}$$

weighting factors:

$$WFL = \left(\frac{3}{2}\right)^2 = \frac{9}{2} = 4.5$$
$$WFU = (10)^2 = 100$$

DETA = 0.05

Problem Requiring Reordering